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A Note on Convergence of Gap Series

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1. Introduction. In the present note we shall assume that $f(t)$ is an integrable function with period 1, such that

$$\int_0^1 f(t)\ dt = 0\ , \qquad \int_0^1 f(t)^2 dt = 1\ ,$$

and $\{n_k\}$ is a sequence of positive integers satisfying $n_{k+1}/n_k > c > 1 (k=1, 2, \cdots)$. M. Kac, R. Salem and A. Zygmund [1] proved the following theorem;

Theorem A. If, for $\alpha > 1$, $f(t)$ satisfies

$$\left(\int_0^1 |f(t)-S_n(t)|^2 dt\right)^{1/2} = O\left((\log\ n)^{-\alpha}\right) \tag{1.1}$$

as $n \to \infty$, then

$$\sum c_k f(n_k t) \tag{1.2}$$

converges a. e., provided that

$$\sum c_k^2 (\log\ k)^2 < \infty, \tag{1.3}$$

where $S_n(t)$ means the n-th partial sum of the Fourier series of $f(t)$. On the other hand if, for $0 < \alpha \leqq 1$, $f(t)$ satisfies (1.1), then

$$\sum \frac{1}{k^\beta} f(n_k t) \tag{1.2'}$$

converges a. e., provided that

$$\beta > 1 - \frac{\alpha}{2}. \tag{1.3'}$$

In Theorem 1 and Theorem 2 of this note, we shall prove that (1.3) and (1.3′) may be replaced by more weaker restrictions respectively. On the other hand S. Izumi [2] obtained a result concerning the special case $\beta = 1$. Our Theorem 2 (iii) contains a generalization of the Izumi's result.

2. Theorem 1. If, for $\alpha>1$, $f(t)$ satisfies (1.1) and for some non-negative integer $\sigma\geqq 0$,

$$\sum c_k^2\ (\log_{\sigma+1}k)^2<\infty \tag{2.1}$$

holds, then (1.2) converges a. e., where $\log_\sigma$ means the σ times iteration of the logarithm

For the proof of this theorem, we need two lemmas.

Lemma 1. If, for $\alpha>0$, $f(t)$ satisfies (1.1), then for any positive integers M and N such as $M<N$, and for any non-decreasing sequence $\{\tau(k)\}$ of integers, we have

$$\int_0^1\left|\sum_M^N c_k\{f(n_kt)-S_{\tau(k)}(n_kt)\}\right|^2dt\leqq F(M,N\,;\,\alpha)\ \sum_M^N c_k^2\Big(\log\tau(k)\Big)^{-\alpha} \tag{2.2}$$

where $F(M,N\,;\,\alpha)$ is defined by

$$\begin{aligned} F(M,N\,;\,\alpha)&=O\Big((N\text{-}M)^{1-\alpha}\Big) && \text{for } 0<\alpha<1,\\ &=O\Big(\log\ (N-M)\Big) && \text{for } \alpha=1,\\ &=O(1) && \text{for } \alpha>1, \end{aligned} \tag{2.3}$$

Proof. If we put, for simplicity of writing, $\varphi_n(t)=f(t)-S_n(t)$, then it is familiar that for $j<k$,

$$\int_0^1\varphi_{\tau(j)}(n_jt)\varphi_{\tau(k)}(n_kt)dt=O\Big((k-j)^{-\alpha}(\log\tau(k))^{-\alpha}\Big)$$

Now from the above relation, we have

$$\begin{aligned}\int_0^1\left|\sum_M^N c_k\varphi_{\tau(k)}(n_kt)\right|^2dt&=\sum_M^N c_k^2\int_0^1\varphi_{\tau(k)}(n_kt)^2dt\\ &+\sum_{k\neq j}c_kc_j\int_0^1\varphi_{\tau(k)}(n_kt)\ \varphi_{\tau(j)}(n_jt)dt\\ &=\sum_M^N c_k^2\,O\Big(\log\tau(k)\Big)^{-2\alpha}+2\sum_{j=M}^{N-1}\sum_{k=j+1}^{N}c_kc_j\,O\Big((k-j)^{-\alpha}(\log\tau(k))^{-\alpha}\Big)\\ &=\sum_M^N c_k^2\,O\Big(\log\tau(k)\Big)^{-2\alpha}+2\sum_{j=M}^{N-1}\sum_{r=1}^{N-j}c_{j+r}c_j\,O\Big(r^{-\alpha}(\log\ \tau(j+r))^{-\alpha}\Big)\\ &<O(1)\sum_M^N c_k^2\Big(\log\ \tau(k)\Big)^{-2\alpha}+O(1)\sum_{r=1}^{N-M}\frac{1}{r^\alpha}\Big(\sum_M^N c_k^2(\log\ \tau(k))^{-\alpha}\Big)\end{aligned}$$

Hence we obtain the lemma.

Lemma 2. Under the hypotheses of Lemma 1, we have

$$\int_0^1\left|\sum_M^N c_k\,S_{\tau(k)}(n_kt)\right|^2dt\leqq F(M,N\,;\,\alpha)\ \sum_M^N c_k^2. \tag{2.4}$$

The Proof is similar as that of Lemma 1.

Remark. In Lemma 1, we may replace $\varphi_{\tau(k)}(n_kt)$ by $\{S_{N(k)}(n_kt)-S_{M(k)}(n_kt)\}$, where $M(k)$ and $N(k)$ are both non-decreasing sequence of integers such as $M(k)$

$<N(k)$, and we have the following inequality,

$$\int_0^1 \left| \sum_M^N c_k\{S_{N(k)}(n_k t)-S_{M(k)}(n_k t)\} \right|^2 dt \leqq F(M,N;\alpha) \sum_M^N c_k^2 \Big(\log M(k)\Big)^{-\alpha}$$

Proof of Theorem 1: Putting, for $k=1, 2, \cdots$,

$$\tau_0(k)=[\exp \lambda_0(k)] \text{ and } \lambda_0(k)=\sqrt{k}\,, \tag{2.5}$$

then we have

$$\sum |c_k| \int_0^1 |\varphi_{\tau_0(k)}(n_k t)|\,dt < \sum |c_k|\; O\Big((\log \tau_0(k))^{-\alpha}\Big) < \infty.$$

This formula indicates the almost everywhere convergence of

$$\sum c_k\, \varphi_{\tau_0(k)}(n_k t), \tag{2.6}$$

Now putting, for $q=1, 2, 3, \cdots$

$$\tau_q(k)=[\exp \lambda_q(k)] \text{ and } \lambda_q(k)=(\log_q k)^{2/\alpha}, \tag{2.7}$$

then for some k_0, $\lambda_q(k)$ is non-decreasing as $k \geqq k_0=k_0(q)$, and we have, for each $q=0, 1, 2, \cdots$, the almost everywhere convergence of

$$\sum c_k\{S_{\tau_q(k)}(n_k t)-S_{\tau_{q+1}(k)}(n_k t)\}. \tag{2.8}$$

For the proof of this, we consider a sequence of trigonometric polynomials

$$X_k(t)=\sum_{l=\nu_k}^{\nu_{k+1}-1} c_l\{S_{\tau_q(l)}(n_l t)-S_{\tau_{q+1}(l)}(n_l t)\} \tag{2.9}$$

where $\{\nu_k\}$ is an increasing sequence of integers such that, for $k_0(q)$

$$\nu_{k_0}=\text{some constant A},\ \nu_{k+1}=\nu_k+\Big[\frac{2}{\log c}\lambda_q(\nu_k)\Big] \tag{2.10}$$

Since the largest frequency of $X_{k-1}(t)$ is $\tau_q(\nu_k)\ n(\nu_k)$ and the smallest frequency of $X_{k+1}(t)$ is $\tau_{q+1}(\nu_{k+1})\ n(\nu_{k+1})$, and moreover we have from the definition of $\{\nu_k\}$

$$\frac{\tau_{q+1}(\nu_{k+1})\ n(\nu_{k+1})}{\tau_q(\nu_k)\ n(\nu_k)} > e > 1 \qquad (k \geqq k_0), \tag{2.11}$$

where $n(k)$ means $n_k (k=1, 2, \cdots)$, so both sequences $\{X_{2k}(t)\,;\, 2k \geqq k_0\}$ and $\{X_{2k-1}(t)\,;\, 2k-1 \geqq k_0\}$ are orthogonal respectively, and by Remark of Lemma 1,

$$\int_0^1 |\sum X_{2k}(t)|^2 dt = \sum \int_0^1 |X_{2k}(t)|^2 dt$$

$$\leqq \sum_k \sum_{l=\nu_{2k}}^{\nu_{2k+1}-1} c_l^2 (\log \tau_{q+1}(l))^{-\alpha} < \sum c_l^2 < \infty.$$

From this and (2.11), $\sum X_{2k}(t)$ is the Fourier serier of some square integrable function and it converges a.e.. By the same way, we obtain the almost everywhere convergence of $\sum X_{2k-1}(t)$ and consequently $\sum X_k(t)$. Now the convergence of (2.8) follows from

$$\sum_k \int_0^1 \max_{\nu_k<m<\nu_{k+1}} \left| \sum_{l=\nu_k}^{m} c_l\{S_{\tau_q(l)}(n_l t)-S_{\tau_{q+1}(l)}(n_l t)\} \right|^2 dt < \infty. \tag{2.12}$$

But this (2.12) is easily verified by use of the Menchov's well known devices. The left hand side of (2.12) is less than by (2.7)

$$\leqq \sum_k \left(\log\ (\nu_{k+1}-\nu_k)\right)^2 \int_0^1 |X_k(t)|^2 dt$$

$$\leqq O(1)\sum_k \left(\log\ \lambda_q(\nu_k)\right)^2 \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l^2 \left(\log\ \tau_{q+1}(l)\right)^{-\alpha}$$

$$\leqq O(1)\sum_k \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l^2 \lambda_{q+1}(l)^{-\alpha}\left(\log\ \lambda_q(l)\right)^2 < O(1)\sum c_k^2$$

As a conclusion of the above arguments, we obtain the following

$$\text{(2.13)}\qquad \left(\begin{array}{l}\text{Under the hypothesis (1.1) and } \sum c_k^2<\infty, \text{ whenever } q \text{ is 0 or a} \\ \text{positive integer, then } \displaystyle\sum_{k=k_0(q)}^{\infty} c_k\{f(n_k t)-S_{\tau_q(k)}(n_k t)\} \\ \text{converges a.e.}\end{array}\right)$$

Lastly if we prove, for some q, the convergence of

$$\text{(2,14)}\qquad \sum c_k S_{\tau_q(k)}(n_k t),$$

then we obtain Theorem 1. Using Lemma 2 instead of Lemma 1, we can easily lead, by the similar methods as the above arguments, for each $q=0, 1, 2,\cdots$ the almost everywhere convergence of

$$\text{(2.15)}\qquad \sum_k \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l S_{\tau_q(l)}(n_l t),$$

and, have, for each $q=0, 1, 2,\cdots$

$$\sum_k \int_0^1 \max_{\nu_k<m<\nu_{k+1}} \left|\sum_{l=\nu_k}^{m} c_l\, S_{\tau_q(l)}(n_l t)\right|^2 dt$$

$$\text{(2.16)}\qquad \leqq \sum_k \left(\log\ (\nu_{k+1}-\nu_k)\right)^2 \int_0^1 \left|\sum_{l=\nu_k}^{\nu_{k+1}-1} c_l S_{\tau_q(l)}(n_l t)\right|^2 dt$$

$$\leqq \sum_k \left(\log\ \lambda_q(\nu_k)\right)^2 \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l^2 \leqq O(1)\sum_l c_l^2(\log_{q+1} l)^2$$

From (2.1), the last series converges when $q=\sigma$. Thus by (2.13), (2.15) and (2.16) we obtain the theorem.

3. Theorem 2. If $f(t)$ satisfies (1.1), then we have,

(i) for $0<\alpha<\frac{1}{2}$ and $\beta>1-\alpha$, (1.2′) converges a.e.,

(ii) for $\frac{1}{2}\leqq\alpha<1$ and $\beta>\frac{1}{2}$, (1.2′) converges a.e.,

(iii) for $\alpha=1$ and $\beta>\frac{1}{2}$

$$\sum \frac{f(n_k t)}{\sqrt{k}(\log\ k)^\beta}$$

converges a. e.

Proof of (i). If α and β satisfy the hypothesis of (i), then we have for each $q=0, 1, 2, \cdots$

$$\frac{\beta}{1-\alpha}>\beta\frac{2-\alpha'^{q}-\alpha'^{q+1}}{1-\alpha'^{q+1}},$$

where $\alpha'=(1-\alpha)/\alpha$, so we determine a number γ such as for each $q=0, 1, 2, \cdots$, σ (where σ is an arbitrary but fixed number)

$$\gamma>1\,, \qquad \frac{\beta}{1-\alpha}>\gamma>\beta\frac{2-\alpha'^{q}-\alpha'^{q+1}}{1-\alpha'^{q+1}}, \tag{3.1}$$

and we define for $q=0, 1, 2, \cdots, \sigma$

$$\lambda_0=k^{(\gamma-\beta)/\alpha},\ \lambda_{q+1}(k)=k^{(\gamma-2\beta)/\alpha}\lambda_q(k)^{\alpha'},\ \tau_q(k)=[\exp\lambda_q(k)]. \tag{3.2}$$

From (3. 2)

$$\lambda_q(k)=k^{\frac{\gamma-2\beta}{\alpha}\frac{1-\alpha'^{q}}{1-\alpha'}+\frac{\gamma-\beta}{\alpha}\alpha'^{q}}$$

holds, and for each q, $\lambda_q(k)$ is non-decreasing as k tends to $+\infty$, and for each $k>1$, and $q=0, 1, 2, \cdots, \sigma-1$, $\lambda_q(k)\geqq\lambda_{q+1}(k)$,

Evidently from (3.2)

$$\sum\frac{1}{l^{\beta}}\int_0^1\left|\varphi_{\tau_0(l)}(n_l t)\right|dt<\infty.$$

follows, and this shows the almost everywhere convergence of

$$\sum_l\frac{1}{l^{\beta}}\{f(n_l t)-S_{\tau_0(l)}(n_l t)\}. \tag{3.3}$$

Using the same formulae as (2.9), (2.10) and (2.11), then from Lemma 2, we have

$$\int_0^1\left|\sum_k X_{2k}(t)\right|^2dt=\sum_k\int_0^1\left|X_{2k}(t)\right|^2dt$$

$$\leqq\sum_k\sum_{l=\nu_{2k}}^{\nu_{2k+1}-1}\frac{1}{l^{2\beta}}\frac{1}{\lambda_{q+1}(l)^{\alpha}}(\nu_{2k+1}-\nu_{2k})^{1-\alpha}$$

$$\leqq\sum_k\sum_{l=\nu_{2k}}^{\nu_{2k+1}-1}\frac{1}{l^{2\beta}}\frac{\lambda_q(l)^{1-\alpha}}{l^{\gamma-2\beta}\lambda_q(l)^{\alpha\alpha'}}=\sum\frac{1}{l^{\gamma}}<\infty.$$

This shows the almost everywhere convergence of $\sum X_{2k}(t)$, and by the similar way, we have the convergence of $\sum X_{2k+1}(t)$ and consequently $\sum X_k(t)$.

Now we shall prove for each $q=0, 1, 2, \cdots, \sigma$.

$$\sum_k\int_0^1\max_{\nu_k<m<\nu_{k+1}}\left|\sum_{l=\nu_k}^{m}\frac{1}{l^{\beta}}\{S_{\tau_q(l)}(n_l t)-S_{\tau_{q+1}(l)}(n_l t)\}\right|^2dt<\infty \tag{3.4}$$

holds. For the left hand side of (3.4) is less than

$$\leqq\sum_k\log(\nu_{k+1}-\nu_k)\sum_{v=0}^{\log(\nu_{k+1}-\nu_k)}\sum_{u=0}^{(\nu_{k+1}-\nu_k)2^{-v}}$$

$$\int_0^1\left|\sum_{l=\nu_k+u2^{v}}^{\nu_k+(u+1)2^{v}-1}\frac{1}{l^{\beta}}\{S_{\tau_q(l)}(n_l t)-S_{\tau_{q+1}(l)}(n_l t)\}\right|^2dt$$

$$\leqq \sum_k \log \lambda_q(\nu_k) \sum_{v=0}^{\log\lambda_q(\nu_k)} \sum_{u=0}^{(\nu_{k+1}-\nu_k)2^{-v}} \sum_{l=\nu_k+u2^v}^{\nu_k+(u+1)2^v-1} \frac{1}{l^{2\beta}} \frac{1}{\lambda_{q+1}(l)^\alpha} 2^{v(1-\alpha)}$$

$$\leqq \sum_l \frac{\log\lambda_q(l)\,\lambda_q(l)^{1-\alpha}}{l^{2\beta}\,l^{\gamma-2\beta}\lambda_q(l)^{\alpha\alpha'}} = \sum \frac{\log l}{l^\gamma} < \infty.$$

Thus by (3.3) and (3.4) we obtain thaat

$$\sum \frac{1}{l^\beta}\{f(n_l t) - S_{\tau_\sigma(l)}(n_l t)\} \tag{3.5}$$

converger a. e.

Now if we repeat nearly the same estimations as (2.15) and (2.16), we can conclude that the almost everywhere convergence of

$$\sum \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t) \tag{3.6}$$

holds, provided that

$$\sum \frac{\log l}{l^{2\beta}\, l^{-(1-\alpha)}\left(\frac{\gamma-2\beta}{\alpha}\frac{1-\alpha'^\sigma}{1-\alpha'} + \frac{\gamma-\beta}{\alpha}\alpha'^\sigma\right)} < \infty. \tag{3.7}$$

For the proof of this, putting

$$X_k(t) = \sum_{l=\nu_k}^{\nu_{k+1}-1} \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t).$$

and

$$\nu_{k+1} = \nu_k + \left[\frac{2}{\log c}\lambda_\sigma(\nu_k)\right], \quad \nu_1 = \text{some constant } A,$$

then we have

$$\int_0^1 \left|\sum_k X_{2k}(t)\right|^2 dt = \sum_k \int_0^1 |X_{2k}(t)|^2 dt$$

$$= \sum_k (\nu_{2k+1} - \nu_{2k})^{1-\alpha} \sum_{\nu_{2k}}^{\nu_{2k+1}-1} \frac{1}{l^{2\beta}} \leqq \sum_k O\Big(\lambda_\sigma(\nu_{2k})\Big)^{1-\alpha} \sum_{\nu_{2k}}^{\nu_{2k+1}-1} \frac{1}{l^{2\beta}}$$

$$= O(1) \sum \frac{1}{l^{2\beta}} \lambda_\sigma(l)^{1-\alpha}.$$

On the other hand, we have

$$\sum_k \int_0^1 \max_{\nu_k < m < \nu_{k+1}} \left| \sum_{l=\nu_k}^m \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t) \right|^2 dt$$

$$\leqq \sum_k \log(\nu_{k+1} - \nu_k) \sum_{v=0}^{\log(\nu_{k+1}-\nu_k)} \sum_{u=0}^{(\nu_{k+1}-\nu_k)2^{-v}} \int_0^1 \left| \sum_{l=\nu_k+u2^v}^{\nu_k+(u+1)2^v-1} \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t) \right|^2 dt$$

$$\leqq \sum_k \log \lambda_\sigma(\nu_k) \sum_v \sum_u \sum_l \frac{1}{l^{2\beta}} 2^{v(1-\alpha)} = \sum_k \log \lambda_\sigma(\nu_k)(\nu_{k+1} - \nu_k)^{1-\alpha} \sum_{l=\nu_k}^{\nu_{k+1}-1} \frac{1}{l^{2\beta}}$$

$$= \sum_k \sum_{l=\nu_k}^{\nu_{k+1}} \frac{1}{l^{2\beta}} \lambda_\sigma(l)^{1-\alpha} \log \lambda_\sigma(l) = \sum_k \sum_{l=\nu_k}^{\nu_{k+1}} \frac{\log l}{l^{2\beta}}\, l^{\left(\frac{\gamma-2\beta}{\alpha}\frac{1-\alpha'^\sigma}{1-\alpha'} + \frac{\gamma-\beta}{\alpha}\alpha'^\sigma\right)(1-\alpha)}$$

These estimations indicate that (3.7) implies the convergence of (3.6).

If we take a sufficiently large σ, then by (3.1)

$$2\beta-(1-\alpha)\left(\frac{\gamma-2\beta}{\alpha}\frac{1-\alpha'^{\sigma}}{1-\alpha'}+\frac{\gamma-\beta}{\alpha}\alpha'^{\sigma}\right)>1, \tag{3.8}$$

consequently (3.7) holds, and (3.6) converges a. e.. From (3.5) and (3.6) the proof of (i) holds.

Proof of (ii). For α and β satisfying the hypotheses of (ii), but $\alpha\neq\frac{1}{2}$ *) we put

$$\gamma>1,\ \beta+\frac{\gamma}{2}>1, \tag{3.9}$$

and we define

$$\lambda_0(k)=k^{\gamma/2\alpha},\ \lambda_{q+1}(k)=\lambda_0(k)^{\alpha'^{q+1}},$$

$$\tau_q(k)=[\exp \lambda_q(k)]. \tag{3.10}$$

The methods of the proof of (ii) are similar as that of (i), so we write only conclusions, that is,

[A] for any $q=0, 1, 2,\cdots$

$$\sum_{l=1}\frac{1}{l^{\beta}}\{f(n_l t)-S_{\tau_q(l)}(n_l t)\} \tag{3.11}$$

converges a.e., provided $2\beta>1$,

[B] for any $q=0, 1, 2,\cdots$

$$\sum_{l=1}\frac{1}{l^{\beta}}S_{\tau_q(l)}(n_l t) \tag{3.12}$$

converges a.e., provided

$$2\beta-\frac{\gamma}{2}\alpha'^{q+1}>1. \tag{3.13}$$

Since $0<\alpha'<1$ holds in our case, for sufficiently large q, (3.13) is satisfied and we obtain the proof of (ii).

Proof of (iii). If we put $\gamma>2\beta$, $\lambda_0(k)=k^{\gamma/2}$, $\lambda_1(k)=(\log k)^{\gamma-2\beta}(\log k)^3$, $\lambda_2(k)=(\log k)^{\gamma-2\beta}(\log\log k)^3$, and $\tau_q(k)=[\exp \lambda_q(k)]$ $(q=0, 1, 2)$, then we can also verify the validity of (iii).

References

(1) M. Kac, R. Salem and A. Zygmund, A Gap Theorem, Trans. Amer. Math. Soc., 63 (1948).

(2) S. Izumi, On the Strong Law of Large Numbers and Gap Series, Tōhoku Math. Jour., 3(1951).

*) For the special case $\alpha=\frac{1}{2}$ and $\beta>\frac{1}{2}$, we put $\varepsilon\leqq(2\beta-1)$, then there exists, for a given $\gamma>1$, a positive integer σ such that $\sigma\varepsilon<\gamma<(\sigma+1)\varepsilon$. If we define $\lambda_q(k)=k^{\gamma-\varepsilon q}$, $\tau_q(k)=[\exp\lambda_q(k)]$ for $q=0, 1,\cdots\cdots\sigma$, then we obtain the result by the similar way as the general case.